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Inverse fuzzy automata and inverse fuzzy languages

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Abstract. In this paper we define an inverse fuzzy automaton and the language recognized by it namely inverse fuzzy language. We prove that a fuzzy language is inverse if and only if the transition monoid of the minimal automaton recognizing that fuzzy language is an inverse monoid or equivalently the syntactic monoid of that fuzzy language is an inverse monoid.

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1. Introduction

The concept of fuzzy sets is introduced by L. A. Zadeh in 1965 [8]. Since then fuzzy set theory is applied to various branches of mathematics and many research works are going on in these areas. W.G Wee[7] introduced the concept of fuzzy automata and Zadeh and Lee[3] generalized the classical notion of languages to the concept of fuzzy languages. It is proved by Petkovic [5] that every monoid is the syntactic monoid of some fuzzy language. Syntactic monoid of a fuzzy language is defined as the transition monoid of the minimal fuzzy automaton recognizing that fuzzy language. The theory of inverse monoids were introduced independently by Wagner and Preston via the study of partial one-one transformations of a set[1]. In this paper we define an inverse fuzzy automaton such that its transition monoid is an inverse monoid and study some of its algebraic properties.

2. Preliminaries

This section present basic definitions and results to be used in the sequel.

Definition 2.1. A semigroup S is said to be an inverse semigroup if for every $a \in S$ there exist a unique $b \in S$ such that $aba = a$ and $bab = b$.

We call b the inverse of a and denote by a^{-1}. If S has an identity then S is said to be an inverse monoid. Inverse monoids form a variety defined by the identities $aa^{-1}a = a$, $aa^{-1}bb^{-1} = bb^{-1}aa^{-1}$. Also $(a^{-1})^{-1} = a$ and $(ab)^{-1} = b^{-1}a^{-1}$. For any element a of an inverse monoid, aa^{-1} is an idempotent and idempotents in an inverse monoid commute. An inverse monoid with a single idempotent is a group. As an analogous to Cayleys theorem for groups, Preston and Wagner proved that an inverse monoid I is isomorphic to a subinverse monoid of the monoid of all one-one partial transformations on I.

Definition 2.2. A fuzzy automaton on an alphabet X is a 5-tuple $M = (Q, X, \mu, i, \tau)$ where Q is a finite set of states, X is a finite set of input symbols and μ is a fuzzy subset of $Q \times X \times Q$ representing the transition mapping, i is a fuzzy subset of Q called initial state, τ is a fuzzy subset of Q called final state.

(Q, X, μ) is called a fuzzy finite state machine. X is called an alphabet and the elements of X are called letters. A word is a finite sequence of letters of X. The length of a word is the number of letters in it. A word of length zero is called the empty word and denoted as Λ. X^+ denote the set of all nonempty words.

X^+ together with the binary operation concatenation is a semigroup called the free semigroup on X and $X^* = X^+ \cup \{\Lambda\}$ is the free monoid on X. A language L is a subset of X^* (see [2, 6]).

A fuzzy automaton can also be represented as a five tuple $(Q, Y, \{T_u|u \in X\}, i, \tau)$ where Y is a subset of Q and $\{T_u|u \in X\}$ is the set of fuzzy transition matrices, $i = [i_1\ i_2\ \dots i_n]$, $i_k \in [0,1]$, $\tau = [j_1\ j_2\ \dots j_n]^T$, $j_k \in [0,1]$, for $k = 1, 2 \dots n$.

μ can be extended to the set $Q \times X^* \times Q$ by

$$\mu(q, \Lambda, p) = \begin{cases} 1, & q = p \\ 0, & q \neq p \end{cases}$$

$$\mu(q, u, p) = \bigvee_{q_i \in Q} \{\mu(q, x_1, q_1) \wedge \mu(q_1, x_2, q_2) \wedge \dots\dots \wedge \mu(q_{k-1}, x_k, p) | x_1 x_2 \dots . x_k = u\}$$

The fuzzy language recognized by this fuzzy automaton is

$$f_M(u) = \bigvee_{q \in Q} \bigvee_{p \in Q} (i(q) \wedge (\mu(q, u, p) \wedge \tau(p)))$$

which can also written as $f_M(u) = i \circ T_u \circ \tau$, where the composition is the max-min composition of fuzzy matrices. The minimal fuzzy recognizer $M(\lambda)$ can be constructed in a way similar to the construction of minimal recognizer for a crisp language. Here $M(\lambda) = (Q, X, \delta, i_0, \tau)$ where $Q = \{\lambda.u | u \in X^*\}$ where $\lambda.u$ is a fuzzy subset of X^* defined by $\lambda.u(w) = \lambda(uw)$ for any $w \in X^*$ and the transitions are defined by $\delta(\lambda.u, w) = \lambda.(uw)$, $i_0 = \lambda.\Lambda$ and $\tau(\lambda.u) = \lambda(u)$.

For a fuzzy automaton $A = (Q, X, \mu, i, \tau)$ define a congruence θ_A on X^* by $u\theta_A v$ if and only if $\mu(q, u, p) = \mu(q, v, p)$, for all $p, q \in Q$. Then the transition monoid $T(A)$ of A is equal to X^*/θ_A.

A fuzzy language λ over an alphabet X is a fuzzy subset of X^*.

Definition 2.3. To each fuzzy language λ over X we associate a congruence P_λ called syntactic congruence as follows.

For $u, v \in X^*$, $uP_\lambda v$ if and only if $\lambda(xuy) = \lambda(xvy)$ for all $x, y \in X^*$.

The quotient monoid (semi group) $Syn(\lambda) = X^*/P_\lambda(X^+/P_\lambda)$ is called the syntactic monoid (semi group) of λ, [4].

Theorem 2.4 ([5]). *For a fuzzy language λ, $T(M(\lambda)) \cong Syn(\lambda)$.*

Let $M = (Q, X, \mu, i, \tau)$ be a fuzzy automaton. We say that the triple (Q, X, μ) is the fuzzy finite state machine associated with M.

Definition 2.5. If $p, q \in Q$, p is called an immediate successor of q if there exist an $a \in X$ such that $\mu(q, a, p) > 0$. p is called a successor of q if there exist $x \in X^*$ such that $\mu(q, x, p) > 0$.

Let $S(q)$ be the set of all successors of q and $T \subseteq Q$. The set of all successors of T, denoted by $S(T) = \cup\{S(q)|q \in T\}$. $N = (T, X, \nu)$ where $T \subseteq Q$, ν is a fuzzy subset of $T \times X \times T$ is a called a submachine of M if $\mu|_{T\times X\times T} = \nu$ and $S(T) \subseteq T$. N is said to be separated if $S(Q - T) \cap T = \phi$. A fuzzy automaton M is said to be connected if it has no separated proper submachines.

3. Inverse fuzzy automata

Definition 3.1. A fuzzy automaton $M = (Q, X, \mu)$ is said to be an inverse fuzzy automaton if for every $x \in X^*$, there exist a unique $y \in X^*$ such that $\mu(q, xyx, p) = \mu(q, x, p)$ and $\mu(q, yxy, p) = \mu(q, y, p)$, for all $p, q \in Q$.

Definition 3.2. A fuzzy language λ is said to be an inverse fuzzy language if it is recognized by an inverse fuzzy automata.

Theorem 3.3. *A fuzzy automaton $A = (Q, X, \mu, i, \tau)$ is inverse if and only if X^*/θ_A is an inverse monoid.*

Proof. We have A is an inverse fuzzy automaton if and only if for each $x \in X^*$ there exist a unique $y \in X^*$ such that for all $p, q \in Q$, $\mu(q, xyx, p) = \mu(q, x, p)$ and $\mu(q, yxy, p) = \mu(q, y, p)$ and so $xyx\theta_A x$ and $yxy\theta_A y$. Then clearly, $[xyx] = [x]$ and $[yxy] = [y]$. Since $[xyx] = [x][y][x]$ and $[yxy] = [y][x][y]$, so we get A is an inverse fuzzy automaton if and only if X^*/θ_A is an inverse monoid. □

Example 3.4. Let $M = (Q, X, \mu, i, \tau)$, where $Q = \{q_0, q_1, q_2\}$, $X = \{a, b\}$, $i = [1\ 0\ 0]$, $\tau = [0\ 0\ 1]^T$ and $\mu : Q \times X \times Q \longrightarrow [0, 1]$ as defined below

$\mu(q_0, a, q_1) = 0.7$, $\mu(q_1, a, q_2) = 0.4$, $\mu(q_2, a, q_0) = 0.3$, $\mu(q_1, b, q_0) = 0.7$, $\mu(q_0, b, q_2) = 0.3$, $\mu(q_2, b, q_1) = 0.4$ and 0 for all other elements of $Q \times X \times Q$.

Then $T_a = \begin{pmatrix} 0 & 0.7 & 0 \\ 0 & 0 & 0.4 \\ 0.3 & 0 & 0 \end{pmatrix}$ $T_b = \begin{pmatrix} 0 & 0 & 0.3 \\ 0.7 & 0 & 0 \\ 0 & 0.4 & 0 \end{pmatrix}$

Also, the transition semigroup is the semigroup generated by $\{T_a, T_b\}$ in which $T_{aba} = T_a$, $T_{bab} = T_b$. So $T(M)$ is an inverse monoid.

Example 3.5. Let $Q = \{q_0, q_1, q_2\}, X = \{a, b\}$. Take $\tilde{X}^*$ as the inverse semigroup generated by X. Consider a deterministic fuzzy automaton $M = (Q, \tilde{X}, \mu)$ over $\tilde{X} = X \cup X^{-1}$. Take a fuzzy subset of $Q \times \tilde{X} \times Q$ such that for every $p \in Q$, there exist a unique $q \in Q$ and an $a \in X$ such that $\mu(q, a, p) > 0$ and $\mu(q, a, p) = \mu(p, a^{-1}, q)$ for all $p, q \in Q, a \in X$. Then M is a connected fuzzy automaton and is inverse. Here $\tilde{X}^*$ acts on Q as one-one partial fuzzy transformations. The transition monoid is a subinverse monoid of the symmetric inverse monoid of all one-one partial fuzzy transformations on Q.

4. Inverse fuzzy languages

We have proved that the transition monoid of an inverse fuzzy automaton is an inverse monoid. If a fuzzy language is recognized by an inverse fuzzy automaton, the corresponding transition monoid should recognize that fuzzy language. A fuzzy language λ over an alphabet X is recognizable by a monoid S if there exists a homomorphism $\phi : X^* \longrightarrow S$ and a fuzzy subset δ of S such that $\lambda = \phi^{-1}(\delta)$ where $\phi^{-1}(\delta)(u) = \delta(\phi(u)$ for all $u \in X^*$. So if λ is an inverse fuzzy language, there exist an inverse monoid I and a fuzzy subset δ of I and a homomorphism ϕ from X^* to I such that $\phi^{-1}(\delta) = \lambda$.

Theorem 4.1. *A fuzzy language λ is inverse if and only if for every $x \in X^*$ there exist a unique $y \in X^*$ such that $\lambda(uxyxv) = \lambda(uxv)$ and $\lambda(uyxyv) = \lambda(uyv)$ for every $u, v \in X^*$.*

Proof. Suppose λ is an inverse fuzzy language. Then there exist an inverse fuzzy automaton M such that $f_M(u) = \lambda(u)$ for all $u \in X^*$. Then the minimal fuzzy automaton $M(\lambda)$ recognizing λ is the quotient automaton with the set of states $\{\lambda.u | u \in X^*\}$ where $\lambda.u$ is a fuzzy subset of X^* defined by $\lambda.u(w) = \lambda(uw)$ and $\lambda.u = \lambda.v$ if and only if $u/R_\lambda = v/R_\lambda$. Also the transition monoid of the quotient fuzzy automaton is isomorphic to the syntactic monoid of the fuzzy language λ. Thus λ is inverse if and only if there exists a minimal fuzzy automaton in which for each $x \in X^*$, there exist a unique $y \in X^*$ such that $\mu(q, xyx, p) = \mu(q, x, p)$ and $\mu(q, yxy, p) = \mu(q, y, p)$ for all p, q in the state set. That is, if and only if $(xyx, x) \in P_\lambda$ and $(yxy, y) \in P_\lambda$. Thus λ is inverse if and only if $\lambda(uxyxv) = \lambda(uxv)$ and $\lambda(uyxyv) = \lambda(uyv)$ for all $u, v \in X^*$ □

Theorem 4.2. *The class of all inverse languages in X^* is closed under finite Boolean operations.*

Proof. Let λ be an inverse fuzzy language on X^*. Then there exist an inverse monoid I, a fuzzy subset δ of I and a homomorphism ϕ from X^* to I such that $\phi^{-1}(\delta)(u) = \lambda(u)$. Then $\overline{\delta}$ will recognize $\overline{\lambda}$ for, $\phi^{-1}(\overline{\delta})(u) = \overline{\delta}(\phi(u)) = 1 - \delta(\phi(u)) = 1 - \phi^{-1}(\delta)(u) = 1 - \lambda(u) = \overline{\lambda}(u)$. Thus $\overline{\lambda}$ is an inverse fuzzy language

To prove the closure property of intersection, let λ_1 and λ_2 be two inverse fuzzy languages on X^*. Then there exist two inverse fuzzy automata $M_1 = (Q_1, X, \mu_1, i_1, \tau_1)$ and $M_2 = (Q_2, X, \mu_2, i_2, \tau_2)$ recognizing λ_1 and λ_2 respectively. Construct a fuzzy automaton $M = (Q_1 \times Q_2, X, \mu_1 \times \mu_2, i_1 \times i_2, \tau_1 \times \tau_2)$ where

$$(\mu_1 \times \mu_2)((q_1, q_2), a, (p_1, p_2)) = \mu_1(q_1, a, p_1) \wedge \mu_2(q_2, a, p_2)$$

for all $a \in X$. $i_1 \times i_2$ and $\tau_1 \times \tau_2$ are fuzzy subsets of $Q_1 \times Q_2$ defined by $(i_1 \times i_2)(q_i, q_j) = i_1(q_i) \wedge i_2(q_j)$ and $(\tau_1 \times \tau_2)(q_i, q_j) = \tau_1(q_i) \wedge \tau_2(q_j), (q_i, q_j) \in Q_1 \times Q_2$. $\mu_1 \times \mu_2$ can be extended to X^* by $(\mu_1 \times \mu_2)((q_1, q_2), x, (p_1, p_2)) = \mu_1(q_1, x, p) \wedge \mu_2(q_2, x, p_2)$ for $x \in X^*$. Then M is an inverse fuzzy automaton since $(\mu_1 \times \mu_2)((q_1, q_2), x, (p_1, p_2)) = \mu_1(q_1, x, p_1) \wedge \mu_2(q_2, x, p_2)$ for all $x \in X^*$ and M_1 and M_2 are inverse fuzzy automata.

The language recognized by M is $\lambda_1 \wedge \lambda_2$. So $\lambda_1 \wedge \lambda_2$ is an inverse fuzzy language.
Closure property of union follows from De-Morgan law. □

5. Homomorphic and inverse homomorphic images of inverse fuzzy automata

Definition 5.1. Let $M_1 = (Q_1, X_1, \mu_1, i_1, \tau_1)$ and $M_2 = (Q_2, X_2, \mu_2, i_2, \tau_2)$ be two fuzzy automata. A pair (α, β) of mappings $\alpha : Q_1 \longrightarrow Q_2$ and $\beta : X_1 \longrightarrow X_2$ is called a homomorphism written as $(\alpha, \beta) : M_1 \longrightarrow M_2$, if $\mu_1(q, x, p) \leq \mu_2(\alpha(q), \beta(x), \alpha(p))$ for all $p, q \in Q_1$ and for all $x \in X_1$. The pair (α, β) is called a strong homomorphism if $\mu_2(\alpha(q), \beta(x), \alpha(p)) = \bigvee\{\mu_1(q, x, t) | t \in Q_1, \alpha(t) = \alpha(p)\}$ for all $q, p \in Q_1$ and for all $x \in X_1$. Also β can be extended to $\beta^* : X_1^* \longrightarrow X_2^*$ by $\beta^*(\Lambda) = \Lambda$ and $\beta^*(ua) = \beta^*(u)\beta^*(a)$ for all $u \in X_1^*$, $a \in X_1$ and $\beta^*(uv) = \beta^*(u)\beta^*(v)$ for all $u, v \in X^*$. If α, β are one-one and onto then (α, β) is called an isomorphism.

Theorem 5.2. *Let M_1, M_2 be two fuzzy automata. Let $(\alpha, \beta) : M_1 \longrightarrow M_2$ be a strong homomorphism. Then α is one-one if and only if $\mu_1(q, x, p) = \mu_2(\alpha(q), \beta^*(x), \alpha(p))$ for all $q, p \in Q$ and $x \in X_1^*$* [4].

Theorem 5.3. *If $M_1 = (Q_1, X, \mu_1)$ and $M_2 = (Q_2, X, \mu_2)$ be two fuzzy automata. Let $(\alpha, \beta) : M_1 \longrightarrow M_2$ be a strong homomorphism and if M_1 is inverse, then $(\alpha, \beta)(M_1)$ is also inverse.*

Proof. Since (α, β) is a strong homomorphism from M_1 to M_2 and M_1 is inverse, so we have $\mu_2(\alpha(q), \beta(x), \alpha(p)) = \bigvee\{\mu_1(q, x, t) : t \in Q,\ \alpha(t) = \alpha(p)\}$ for all $p, q \in Q_1$, $x \in X$ and for every $x \in X^*$ there exist a unique $y \in X^*$ such that $\mu_1(q, xyx, p) = \mu_1(q, x, p)$ and $\mu_1(q, yxy, p) = \mu_2(q, y, p)$.

$\mu_2(\alpha(q), \beta^*(x)\beta^*(y)\beta^*(x), \alpha(p)) = \mu_2(\alpha(q), \beta^*(xyx), \alpha(p)) = \bigvee\{\mu_1(q, xyx, t) : t \in Q, \alpha(t) = \alpha(p)\} = \bigvee\{\mu_1(q, x, t) : t \in Q, \alpha(t) = \alpha(p)\} = \mu_2(\alpha(q), \beta^*(x), \alpha(p))$.
And,
$\mu_2(\alpha(q), \beta^*(y)\beta^*(x)\beta^*(y), \alpha(p)) = \mu_2(\alpha(q), \beta^*(yxy), \alpha(p)) = \bigvee\{\mu_1(q, yxy, t) : t \in Q, \alpha(t) = \alpha(p)\} = \bigvee\{\mu_1(q, y, t) : t \in Q, \alpha(t) = \alpha(p)\} = \mu_2(\alpha(q), \beta^*(y), \alpha(p))$.
Thus the image of M_1 under (α, β) is an inverse fuzzy automata.

If f_{M_1} is a fuzzy language recognized by M_1, then its image $\beta(f_{M_1})$ defined as

$$\beta(f_{M_1}(u)) = \begin{cases} \bigvee f_{M_1}(v) : \beta^*(v) = u \text{ if } {\beta^*}^{-1}(u) \neq \phi\} \\ 0 \text{ otherwise} \end{cases}$$

is recognized by $(\alpha, \beta)(M_1)$. So the image of f_{M_1} is inverse since the image automaton is inverse. □

Theorem 5.4. *If $M_1 = (Q_1, X, \mu_1)$ and $M_2 = (Q_2, X, \mu_2)$ be two fuzzy automata. Let $(\alpha, \beta) : M_1 \longrightarrow M_2$ be a strong homomorphism with α, β being one-one and onto and if M_2 is inverse, then $(\alpha, \beta)^{-1}(M_2)$ is also inverse.*

Proof. Suppose (α, β) be a strong homomorphism with α, being one-one onto. Then $(\alpha, \beta) : M_1 \longrightarrow M_2$ has the property $\mu_2(\alpha(q), \beta^*(x), \alpha(p)) = \mu_1(q, x, p)$ for all $x \in X^*$.[4]

Let M_2 be an inverse fuzzy automata. Then for every $x \in X^*$, there exists a unique $y \in X^*$ such that $\mu_2(q, xyx, p) = \mu_2(q, x, p)$ and $\mu_2(q, yxy, p) = \mu(q, y, p)$ for all $q, p \in Q_2$.
That is,

$$\mu_1(\alpha^{-1}(q), {\beta^*}^{-1}(xyx), \alpha^{-1}(p)) = \mu_1(\alpha^{-1}(q), {\beta^*}^{-1}(x), \alpha^{-1}(p))$$

and

$$\mu_1(\alpha^{-1}(q), {\beta^*}^{-1}(yxy), \alpha^{-1}(p)) = \mu_1(\alpha^{-1}(q), {\beta^*}^{-1}(y), \alpha^{-1}(p))$$

which implies

$$\mu_1(\alpha^{-1}(q), {\beta^*}^{-1}(x){\beta^*}^{-1}(y){\beta^*}^{-1}(x), \alpha^{-1}(p)) = \mu_1(\alpha^{-1}(q), {\beta^*}^{-1}(x), \alpha^{-1}(p))$$

and

$$\mu_1(\alpha^{-1}(q), {\beta^*}^{-1}(y){\beta^*}^{-1}(x)\beta^{*-1}(y), \alpha^{-1}(p)) = \mu_1(\alpha^{-1}(q), {\beta^*}^{-1}(y), \alpha^{-1}(p)).$$

Thus, $(\alpha, \beta)^{-1}(M_2)$ is inverse.

If f_{M_2} is a fuzzy language recognized by M_2, then its inverse image $\beta^{-1}(f_{M_2})$ defined as $\beta^{-1}(f_{M_2})(u) = f_{M_2}(\beta(u))$ is recognized by $(\alpha, \beta)^{-1}(M_2)$ and so $\beta^{-1}(f_{M_2})$ is inverse since the inverse image automaton is inverse. □

6. Conclusions

In this paper we have defined an inverse fuzzy automaton and the fuzzy language recognized by it namely inverse fuzzy language. We proved that the transition monoid of an inverse fuzzy automaton is an inverse monoid. We also proved some algebraic properties of the class of inverse fuzzy languages such as it is closed under finite Boolean operations, homomorphic and inverse homomorphic images.

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